1 Poisson Processes

An *arrival* is simply an occurance of some event – like a phone call, a job offer, or whatever – that happens at a particular point in time. We want to talk about a class of continuous time stochastic processes called *arrival processes* that describe when and how these events occur.

Let Ω be a sample space and P a probability. For any $\omega \in \Omega$, for all t we define $N_t(\omega)$ as the number of arrivals in the time interval $[0, t]$ given the realization ω. We call $N = \{N_t, t \geq 0\}$ an arrivial process. Clearly, as time evolves $N_t(\omega)$ jumps up by integer amounts with new arrivals. The type of arrivial process we are most interested in is called a *Poisson process*. A Poisson process is defined as an arrival process that satisfies the following three axioms:

1. for almost all ω, each jump is of size 1;
2. for all $t, s > 0$, $N_{t+s} - N_t$ is independent of the history up to t, $\{N_u, u \leq t\}$;
3. for all $t, s > 0$, $N_{t+s} - N_t$ is independent of t.

The first axiom says that there is a zero probability of two arrivals at the exact same instant in time; the second says that the number of arrivals in the future is independant of what happened in the past; and the third says the number of arrivals in the future is indentically distributed over time, or that the process is stationary. What is interesting is that these simple qualitative feeatures of the process imply the following:

Lemma 1 *If N is a Poisson process then for all $t \geq 0$, $P(N_t = 0) = e^{-\lambda t}$ for some $\lambda \geq 0$.*

Proof. *By the independence axiom, $P(N_{t+s} = 0) = P(N_t = 0)P(N_{t+s} - N_t = 0)$. By the stationarity axiom, $P(N_{t+s} - N_t = 0) = P(N_s = 0)$. Hence, if we let $f(t) = P(N_t = 0)$, we have just established*

$$f(t + s) = f(t)f(s).$$

It is known that the only possible solution to this functional equation is either $f(t) = 0$ for all t, or $f(t) = e^{-\lambda t}$ for some $\lambda \geq 0$. The former possibilty *contradicts the first axiom in the definition of a Poisson process, since if there is* 0 *probability of* $N_t = 0$ *no matter how small is* t*, then we must have an infinite number of arrivals in any interval. So we are left with the second possibility, which was what we set out to show.* ■

With a Poisson process, the probability of an arrival at a certain fixed point in time is zero. The next result says more: as the length of an interval t shrinks to 0 the probability of an arrival divide by t goes to a constant λ. Also, the probability of more than 1 arrival divided by t goes to 0. This is often written $P(N_t \geq 2) = o(t)$, where $o(t)$ is the standard notation for a function with the property that $\frac{o(t)}{t} \to 0$ as $t \to 0$.

Lemma 2 *If* N *is a Poisson process then* $\lim_{t\to 0} \frac{P(N_t=1)}{t} = \lambda$ *and* $\lim_{t\to 0} \frac{P(N_t\geq 2)}{t} = 0$.

The next result gives the exact formula for the probability distribution of N_t.

Theorem 3 *If* N *is a Poisson process then the number of arrivals in an interval of length* t *has a Poisson distribution with parameter* λt*; that is* $P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$.

Proof. Define the function

$$G(t) = E\left[\alpha^{N_t}\right] = \sum_{n=0}^{\infty} \alpha^n P(N_t = n).$$

for some α. Using the independence axiom in the definition of a Poisson process

$$E\left[\alpha^{N_{t+s}}\right] = E\left[\alpha^{N_t}\alpha^{N_{t+s}-N_t}\right] = E\left[\alpha^{N_t}\right] E\left[\alpha^{N_{t+s}-N_t}\right].$$

In other words, $G(t+s) = G(t)G(s)$. Since $G(t) \neq 0$, we know $G(t) = e^{tg(\alpha)}$ for all $t \geq 0$. Notice $g(\alpha)$ is the derivative of $G(t)$ at $t = 0$; that is

$$g(\alpha) = \lim_{t\to 0} \frac{G(t) - G(0)}{t}.$$

Straightforward analysis (see Cinlar) allows us to simplify the limit on the RHS to $-\lambda + \lambda\alpha$. Hence we have $G(t) = e^{-\lambda t + \lambda t \alpha} = e^{-\lambda t} e^{\lambda t \alpha}$, or using the definition of e,

$$G(t) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t \alpha)^k}{k!}.$$

Combining this with the definition of $G(t)$, we have

$$\sum_{k=0}^{\infty} \alpha^k P(N_t = k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t} (\lambda t)^k \alpha^k}{k!}.$$

Equality of these summations for all α implies equality of the terms, or $P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$, and this completes the proof. ■

So far in the analysis λ is just a constant. Notice however that the previous result imples

$$E\left[N_t\right] = \sum_{k=0}^{\infty} k P(N_t = k) = \sum_{k=0}^{\infty} k \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \lambda t,$$

using $e^{-\lambda t} = \sum \frac{(-\lambda t)^k}{k!}$. Hence, λ is expected number of arrivals per period (i.e., in an interval of length 1), and is therefore called the *arrival rate*.

The next result characterizes Poisson processes using conditions different from the axioms in the definition, and therefore could be interpretted as an alternative, equivalent, definition. What is perhaps surprising is that condition (ii) is sufficient to yield both the independence and stationarity axioms in the definition.

Theorem 4 *N is a Poisson process iff (i) for almost all ω, each jump of N is of size 1; and (ii) for all $t, s \geq 0$ $E\left[N_{t+s} - N_t | N_u, u < t\right] = \lambda s$.*

For the next result, let T denote a collection of disjoint intervals of time (e.g., Tusday 10am-noon and Saturday all day). Then the probability of the number of arrivials in T depends only on the total time involved.

Theorem 5 *N is a Poisson process iff $P(N_T = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$ for any subset T which is the union of a finite number of intervals with total length t.*

Note that $\hat{\lambda} = N_t/t$ is a good estimate of the arrival rate, in the sense that $\lim_{t\to\infty} N_t/t = \lambda$. In practice, the estimate is quite good as long as $\lambda t \geq 10$.

We now introduce arrival times. Let T_j be the time of the j^{th} arrival. Above we saw that the probability of no arrivals in the interval $[t, t+s]$ is given by $e^{-\lambda s}$ where λ is the arrival rate. Similarly, $P\left[N_{T_n+s} - N_{T_n} = 0\right] = e^{-\lambda s}$, since the event in brackets is simply the event that no arrivals occur in an interval of length s stating at the time of the (random) n^{th} arrival, T_n. Notice that

$$P\left[T_{n+1} - T_n \leq s | T_0, T_1, ...T_n\right] = 1 - e^{-\lambda s},$$

since the event in question is simply the event that at least one arrival does occur between T_n and $T_n + s$. This says that the interarrival times T_1, $T_2 - T_1$, $T_3 - T_2$,... of a Poisson process are i.i.d. with distribution function (CDF) given by $1 - e^{-\lambda s}$.

This CDF has density $\lambda e^{-\lambda t}$ and is called the exponential distribution. So we see that if N is a Poisson process then the interarrival times are i.i.d. with an exponential distribution; the converse is also true (so that again we have an alternative, equivalent definition of a Poisson process. An interesting thing about this result is that the exponential density is monotone decreasing; hence there is a high probability of a short interval and a small probability of a long interval between arrivials. This means that a typical realization will have lots of arrivals bunched together spaced out by long but rare intervals with no arrivals. To the untrained eye this will look like arrivals come in "streaks" but of course they do not.

In particular, the exponential distribution is what we call *memoryless*:

$$P\left[x > t + s | x > t\right] = P\left[x > s\right].$$

So, for example, if the bus arrives according to a Poisson process, it does not matter how long you have already waited for the time when the next bus arrives. In particular, the expected time until the next arrival is constant and given by $1/\lambda$. More generally, the expected time for n arrivals to occur is always n/λ, regarless of the history of arrivals.